The precession equations (u, v, w correspond to α, β, γ) are

$$ku^{\cdot} + Hg_1v^{\cdot} + Hg_2w^{\cdot} = 0,$$
$$kv^{\cdot} - Hg_1u^{\cdot} + Hg_3w^{\cdot} = 0, \qquad (12)$$
$$kw^{\cdot} - Hg_2u^{\cdot} - Hg_3v^{\cdot} = 0$$

and for $k \neq 0$ and $\mu = H^{-1} \neq 0$ have the unique solution

$$u = \alpha_0, \quad v = \beta_0, \quad w = \gamma_0 \qquad (13)$$

If we divide Eqs. (11) by H, introduce the small parameter $\mu = H^{-1}$, integrate them, and retain only the principal terms in the general solution, then we have

$$\alpha = \alpha_0 + g_3E, \ \beta = \beta_0 - g_2E, \ \gamma = \gamma_0 + g_1E$$

where

$$E = \frac{g_3\alpha_0^{\cdot} - g_2\beta_0^{\cdot} + g_1\gamma_0^{\cdot}}{k(g_1^2 + g_2^2 + g_3^2)} \ (1 - e^{-kt})$$

This solution differs from solution (13) by terms nondepending on the small parameter $\mu = H^{-1}$; therefore, the passage from the full equations (11) to the precession equations (12) is inadmissible (in the example given $\det G = 0$).

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INVARIANTS OF MULTIDIMENSIONAL SYSTEMS WITH ONE RESONANCE RELATION

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L. M. MARKHASHOV
(Moscow)

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The description of invariants generated in systems of ordinary equations by homeomorphisms of a neighborhood of a singular point is connected both with stability problems [1, 2] as well as with the broader problems of the topological, analytical (or formal) classification of such systems [3, 4]. If the eigenvalues of the system's linear part are related by only one resonance relation, a reduction to normal form [5] enables us to extend the results obtained in [6] to invariants of an nth-order system [7]. Namely, we have shown that the group of all analytic

homeomorphisms of a neighborhood of a singular point generates in the equations' coefficient space nh invariant sets depending upon the first $2qh+1$ terms of the expansion of the right-hand sides (q is the order of the resonance, h is the codimension of the system's degeneracy). Besides these the group can have only singular invariant sets (depending on all the system's coefficients).

1. Formulation of results. We examine nth-order autonomous systems

$$\dot{x} = f(x), \qquad f(0) = 0 \tag{1.1}$$

Here $f(x)$ is a vector-valued functions analytic in the neighborhood of the point $x=0$. The eigenvalues λ_i of the linear part are related by the single resonance relation

$$\lambda_1 n_1 + \ldots + \lambda_n n_n = 0 \tag{1.2}$$

There exists a unique formal power series

$$u = u_q + u_{q+1} + \ldots, \qquad u_q = x_1^{n_1} \ldots x_n^{n_n} \tag{1.3}$$

satisfying the conditions:

1) resonance terms are absent in the difference $u - u_q$;

2) $Lu \equiv \sum_{i=1}^{n} f_i(x) \dfrac{\partial u}{\partial x_i} = g_{h+1} u_q^{h+1} + g_{h+2} u_q^{h+2} + \ldots, \qquad g_{h+1} \neq 0$

The number $h \geqslant 1$ is called the codimension of the degeneracy of system (1.1). The sign of the number g_{h+1} determines the stability of the point $x=0$ in the critical cases of one zero root or of a pair of pure imaginary roots.

The group G of all analytic homeomorphisms of a neighborhood of point $x=0$ generates a system of invariant sets in the space of coefficients of the expansion of $f(x)$. Let ρ_s be the number of those of them which depend only on terms of order no higher than s in the expansion of $f(x)$. The number ρ_s does not decrease as s grows. However, the following statement is valid.

Theorem. The number $\rho_{s_0} = \max\limits_{s<\infty} \rho_s$ of invariant sets depending only of a finite segment of $f(x)$ is finite

$$\rho_{s_0} = nh$$

while the maximum order s_0 of this segment is determined by the formula

$$s_0 = 2qh + 1$$

These exhaust all invariant sets of formal transformations. Besides them the analytic group G can have only singular invariant sets (depending on all coefficients of the expansion of $f(x)$) responsible for the convergence of the transformations.

2. Proof of the theorem. We do not detail the presentation of the stages in the proof because they are analogous to those in [6]. For an arbitrary power series $\xi_k = \Sigma c_{k_1 \ldots k_n}^{(k)} x_1^{k_1} \ldots x_n^{k_n}$ and for the operator $Z = \Sigma \xi_k \partial / \partial x_k$ we set

$$\xi_k^{\mu} = \sum_{k_1\lambda_1 + \ldots + k_n\lambda_n = \mu} c_{k_1 \ldots k_n}^{(k)} x_1^{k_1} \ldots x_n^{k_n}, \qquad Z^{\nu} = \sum_k \xi_k^{\nu+\lambda_k} \frac{\partial}{\partial x_k}$$

2.1. Since a formal transformation of any analytic system to a normal form always exists [5], the problem is equivalent to the classification of normal forms relative to the group G of transformations preserving them. For an arbitrary element of Z of the corresponding algebra $[L, Z] = 0$. In particular $[L, Z]^{\nu} \equiv [L^{\circ}, Z^{\nu}] = 0$. Hence

$$[L_1, Z_l^{\nu}] = - \sum_{\alpha+\beta=l+1} [L_{\alpha}, Z_{\beta}^{\nu}]$$

Since

$$[L_1, Z_l^{\nu}] = \nu Z_l^{\nu} \ (L_1 \equiv \Sigma \lambda_k x_k \partial / \partial x_k)$$

$Z_l^{\nu} = 0$ follows from $Z_1^{\nu} = \ldots = Z_{l-1}^{\nu} = 0$ when $\nu \neq 0$. Therefore, a normal form is preserved only by transformations with operators of the form $Z = Z^{\circ}$.

2. 2. Series (1. 3) is determined by the conditions $u^{\circ} = u_q$, $(Lu)^{\nu} = 0$ for all $\nu \neq 0$. Under the action of the normalizing transformation $x = x' \ (1 + \omega(x'))$, $\omega(0) = 0$, the series u and the operator L are transformed, respectively, to u' and to the operator

$$L^{\circ} \quad L_1 + \overset{\circ}{L}_{qm+1} + \ldots + \overset{\circ}{L}_{q(m+1)+1} + \ldots$$

in the normal form. Here $Lu = L^{\circ}u'$. We obtain

$$(Lu)^{\nu} = (L^{\circ}u')^{\nu} = L^{\circ}u'^{\nu} = 0$$

It is easy to find $u' = u'^{\circ}$, where in the new variables $u'^{\circ} = u_q + \ldots$. Further,

$$L^{\circ}u' = (L^{\circ}u')^{\circ} = (Lu)^{\circ} = g_{h+1}u_q^{h+1} + \ldots |_{x \to x'} = g_{h+1}u_q^{h+1} + \ldots$$

Hence we see that the numbers h and g_{h+1} are preserved under a normalizing transformation. Furthermore,

$$m \leqslant h \tag{2.1}$$

$$L^{\circ}u_q = \overset{\circ}{L}_{q(h+1)}u_q + \ldots = g_{h+1}u_q^{h+1} + \ldots \tag{2.2}$$

2. 3. The operators we encounter subsequently form a series composed of operators of the form

$$Z_{q\mu+1} = u_q^{\mu} \left(\alpha_{\mu,1} x_1 \frac{\partial}{\partial x_1} + \ldots + \alpha_{\mu,n} x_n \frac{\partial}{\partial x_n} \right) \equiv u_q^{\mu} Z_1(\alpha_{\mu}) \tag{2.3}$$

If $Z_1(\alpha_{\mu})\, u_q \neq 0$, then $Z(\alpha_{\mu})\, u_q = \beta u_q$, $\beta = \alpha_{\mu,1}n_1 + \ldots + \alpha_{\mu,n}n_n$ and the expansion

$$Z_1(\alpha_{\mu}) = Z_1(\alpha_{\mu}') + \frac{\beta}{q} X_1$$

$$Z_1(\alpha_{\mu}')\, u_q = 0, \qquad X_1 = x_1 \frac{\partial}{\partial x_1} + \ldots + x_n \frac{\partial}{\partial x_n}$$

holds. Consider n linearly independent operators

$$X_1, Z_1(\alpha_1), \ldots, Z(\alpha_{n-1}) \qquad (Z_1(\alpha_k)\, u_q = 0) \tag{2.4}$$

Any operator of form (2. 3) can be written as a linear combination of them

$$Z_{q\mu+1} = u_q^{\mu} (\beta_0 X_1 + \beta_1 Z_1(\alpha_1) + \ldots + \beta_{n-1} Z_1(\alpha_{n-1}))$$

Every operator $Z_{(\mu)} \equiv Z_{q\mu+1} + Z_{q(\mu+1)+1} + \ldots$ satisfies the identity

$$[L, Z_{(\mu)}] = [L, Z]^{\circ}_{pq+1} + [L, Z]^{\circ}_{q(p+1)+1} + \ldots$$

for some positive integer p. If the operators $Z_{q(\mu+1)+1}$, $Z_{q(\mu+2)+1}, \ldots$ are chosen such that the number p is maximal for the specified $Z_{q\mu+1}$, the operator $Z_{(\mu)}$ is said to be maximal and a positive integer $\tau = q(p-\mu)$ is associated to it. The next problem is to compute the numbers τ for all maximal operators whose formal expansions start with operators (2.4) multiplied by u_q^{μ}, $\mu = 0, 1, \ldots$ For operators $Z_1(\alpha_k)$ we trivially obtain $[L, Z_1(\alpha_k)] = 0$, and, hence, $\tau = \infty$.

2.4. Consider the operator $X = X_1 + \overset{\circ}{X}_{q+1} + \overset{\circ}{X}_{2q+1} + \ldots$ We have

$$[L, X]_{kq+1} = 0, \qquad k < m$$

$$[L, X]_{mq+1} = [L_1, \overset{\circ}{X}_{mq+1}] + [\overset{\circ}{L}_{mq+1}, X_1] = -mq\overset{\circ}{L}_{mq+1}$$

Consequently, $[L, X] = -mq\overset{\circ}{L}_{mq+1} + \ldots$ independently of the choice of the operators $\overset{\circ}{X}_{kq+1}$, $k \geqslant 1$, so that $\tau = qm$ for the operator X. For operators of the form $X_{(\mu)} = u_q^{\mu} X_1 + \ldots$ we have $\tau = qm$. In fact, independently of the choice of $\overset{\circ}{X}_{q(\mu+1)+1}, \ldots$, $[L, X_{(\mu)}] = [\overset{\circ}{L}_{mq+1}, u_q^{\mu} X_1] + \ldots = -mqu_q^{\mu} L_{mq+1} + \ldots$

2.5. Let us compute τ for operators of the form

$$Z_{(\mu)} = u_q^{\mu}\left(\alpha_1 x_1 \frac{\partial}{\partial x_1} + \ldots + \alpha_n x_n \frac{\partial}{\partial x_n}\right) \ldots, \qquad \alpha_1 n_1 + \ldots + \alpha_n n_n = 0$$

We denote

$$L = \varphi_1(u)\, x_1 \partial / \partial x_1 + \ldots + \varphi_n(u)\, x_n \partial / \partial x_n \tag{2.5}$$

$$\varphi_k(u) = \lambda_k + a_{km} u^m + \ldots$$

$$(a_{1l} n_1 + \ldots + a_{nl} n_n = 0, \quad l < h \quad \text{by virtue of (2.2)}$$

$$Z_{(\mu)} = \psi_1(u)\, x_1 \partial / \partial x_1 + \ldots + \psi_n(u)\, x_n \partial / \partial x_n$$

$$\psi_k(u) = \alpha_{k\mu} u^{\mu} + \ldots$$

We obtain

$$[L, Z_{(\mu)}] = \sum_l (\varphi\psi_l' - \psi\varphi_l')\, u_q x_l \frac{\partial}{\partial x_l} \equiv \sum_l u_q x_l \Phi_l \frac{\partial}{\partial x_l}$$

$$\varphi = n_1\varphi_1 + \ldots + n_n\varphi_n, \qquad \psi = n_1\psi_1 + \ldots + n_n\psi_n$$

$$\sum_k n_k \Phi_k = \varphi\psi' - \psi\varphi'$$

Let $\sigma(f)$ denote the lowest power appearing in the expansion of f in a power series (in u). From equality (2.5) it follows that

$$\sigma(u\Phi_k) \leqslant \mu + 2h - m \quad \text{for} \quad \mu \neq m$$

In fact,

$$\sigma(\varphi) = h,\ \sigma(\varphi') = h - 1,\ \sigma(\Phi_k) \geqslant \mu + h$$

There exists k such that $\sigma(\varphi\psi_k' + \Phi_k) = h + \mu - 1$. Further,

$$\sigma(\psi) = \sigma\left(\frac{\varphi\psi_k' + \Phi_k}{\varphi_k'}\right) \leqslant h - m + \mu$$

Hence, by virtue of the condition $\mu \neq m$

$$\sigma(\Phi_k) \leqslant \sigma\left(\sum_k n_k \Phi_k\right) = \sigma(\psi\varphi' - \varphi\psi') = 2h - m + \mu - 1$$

Consequently, $\sigma(u\Phi_k) \leqslant \mu + 2h - m$.

Let $\psi = cu^{\mu+h-m} + \ldots$. In order to obtain $\sigma(u\Phi_k) = \mu + 2h - m$ it is necessary that the equalities $\psi\varphi_k' - \varphi\psi_k' = 0,\ k = 1, \ldots, n$ be fulfilled up to order $2h - m - 1$ inclusive. In particular, the conditions

$$mca_{km} - \mu b\alpha_{km} = 0 \quad (b = a_{1h}n_1 + \ldots + a_{nh}n_n)$$

must be fulfilled. In other words, the operator for which $\tau = q(2h - m)$ is determined uniquely. For other operators, being linearly independent with the ones indicated, the equalities $\alpha_{km} = \lambda a_{km}$ cannot be fulfilled for any λ whatsoever. Consequently, for these operators $mca_{km} - \mu b\alpha_{km} \neq 0$ and $\tau = qh$.

The functions Φ_k of maximal power $\mu + 2h - m$ are obtained in the following way. We define ψ_k by the conditions

$$u^{\mu-m}\varphi_1' - \psi_1' = (m - \mu)\,u^{\mu-m-1}\varphi, \qquad u^{\mu-m}\varphi_k' - \psi_k' = 0, \qquad k \geqslant 2$$

Then $u^{\mu-m}\varphi' - \psi' = (m - \mu)\ u^{\mu-m-1}\varphi$, whence $\psi = u^{\mu-m}\varphi$. Therefore,

$$\psi\varphi_k' - \varphi\psi_k' = \varphi\,(u^{\mu-m}\varphi_k' - \psi_k') = \delta_k^1\,(m - \mu)\,u^{\mu-m-1}\varphi^2$$

i.e. $\sigma(\psi\varphi_k' - \varphi\psi_k') = \mu + 2h - m - 1,\ k \geqslant 1$. Thus, when $\mu \neq m$ we have $\tau = qh$ for all operators $Z_{(\mu)} = u_q^{\mu}\,Z_1 + \ldots$ except one (which we denote $Y_{(\mu)}$). For the operator $Y_{(\mu)}$ we have $\tau = q(2h - m)$.

2.6. Consider the case $\mu = m$.

2.6.1. If we choose $\psi_k' = \varphi_k',\ k \geqslant 1$, then $\Phi_k = 0,\ k \geqslant 1$, so that $\tau = \infty$ for the corresponding operator $(Y_{(m)})$.

2.6.2. If $\psi_k' \neq \varphi_k'$, then the first terms of the expansion of ψ_k' and φ_k' cannot coincide since this leads to the maximal operator already considered in Sect. 2.6.1. Hence it follows that $\sigma(\psi\varphi_k' - \varphi\psi_k') = \mu + h - 1$ and, hence $\tau = qh$.

2.7. An ordered set of coefficients a of polynomials of fixed degree s, being segments of expansions of $f(x)$, can be treated as coordinates of points of the Euclidean space R_s. We assume that the order ratio for R_s and R_{s+k} on a coinciding set of elements is the same. The infinite-dimensional linear space R of all coefficients can be considered as the inductive limit of the sequence $R_2, R_3, \ldots$.

The group G of all analytic transformations of a neighborhood of point $x = 0$, leaving this point in place and preserving the linear part of system (1.1), induces a group of transformations G' in $R : G' \times R \to R$. The spaces R are invariant relative to the transformations from G', while the collection of transformations from G' acting non-identically in R forms a Lie group G_s'. Let

$$Z = \sum_i \xi_i(x)\,\frac{\partial}{\partial x_i}, \qquad Z^* = Z + \sum_k \zeta_k(a)\,\frac{\partial}{\partial a_k}$$

be operators corresponding to one-parameter subgroups of groups G and $G \times G'$. The condition for the invariance of system (1.1) relative to the transformations from group $G \times G'$ yields $[L, Z^*] = 0$, or equivalently

$$[L, Z] = \sum_{i,k} \zeta_k(a)\,\frac{\partial f_i}{\partial a_k}\,\frac{\partial}{\partial x_i} \tag{2.6}$$

Equality (2.6) is fulfilled identically with respect to x and serves for the computation of the elements $\zeta_k^j(a)$ of the vector matrix (ζ_k^j) of the algebra corresponding to group G'. This matrix has a block-triangular structure. If operator Z is maximal, then in addition, all elements of its rows belonging to $\tau = q(p - \mu)$ nonzero blocks, vanish. Here it is impossible to increase this number with any linear combination of operator Z with higher-order operators. The number ρ_s of invariant sets generated by group G' in space R_s is determined by the number of zero rows in the corresponding matrix $(\zeta_k^j)_s$, i.e. by the number of maximal operators $Z_{(\mu)}$ for which simultaneously

$$qp + 1 > s, \qquad q\mu + 1 \leqslant s \tag{2.7}$$

If system (1.1) is written in normal form from the very start, then $s = qs^* + 1$, $s^* = 0, 1, \ldots$. Setting $\tau^* = p - \tau$, we write inequalities (2.7) as

$$s^* - \tau^* < \mu \leqslant s^* \tag{2.8}$$

Let r_1, r_2, r_3 be the number of zero rows generated in the matrix $(\zeta_k^j)_s$ by the operators for which $\tau^* = m, h, 2h - m$, respectively. From inequalities (2.8), with due regard to the preceding results, we obtain

$$r_1 = m \qquad (s^* \geqslant m) \qquad\qquad r_3 = \begin{cases} 2h - m & (2h - m \leqslant s^* < 2h) \\ 2h - m + 1 & (s^* \geqslant 2h) \end{cases}$$
$$r_2 = (n-2)h \qquad (s^* \geqslant h)$$

In the computation of r_3 we have taken into account that although $\mu > s^* - \tau^* \geqslant 2h - (2h - m) = m$, for $s^* \geqslant 2h$, among the operators satisfying inequalities (2.8) we should include one more, namely, $Y_{(m)}$. The number of invariant sets is computed from the formula $\rho_s = r_1 + r_2 + r_3 - 1$ (the similarity transformation, not taken into account above, decreases the number of invariant sets by unity). Hence $\rho_{s_0} = nh$. From the formulas for r_i we see that this number ceases to increase when $s^* \geqslant 2h$. Consequently, $s_0 = 2qh + 1$.

3. Example. We examine a fourth-order system in normal form with a degeneracy codimension $h = 1$

$$\dot{x_i} = x_i(\lambda_i + a_{i1}u + a_{i2}u^2 + \ldots), \quad i \leqslant 4 \tag{3.1}$$
$$u = x_1^{n_1} \ldots x_4^{n_4}, \quad \lambda_1 n_1 + \lambda_2 n_2 + \lambda_3 n_3 + \lambda_4 n_4 = 0$$

According to the theorem in this paper, system (3.1) has four invariants depending on segments of the right-hand sides of order not higher than $2q + 1$, i.e. on the coefficients $a_{11}, \ldots, a_{42}$. Let us find these invariants.

The components of the operators corresponding to one-parameter groups preserving the normal form of Eqs. (3.1) have the form (only the transformations affecting coefficients $a_{11}, \ldots, a_{42}$) are considered)

$$\xi_i = \alpha_i x_i u$$

From the defining equations

$$\alpha_i x_i \sum_{j=0}^{\infty} (a_{ij} + n_1 a_{1j} + \ldots + n_n a_{nj})\, u^{j+1} = \alpha_i x_i \sum_{j=0}^{\infty} a_{ij} u^{j+1} +$$
$$x_i \sum_{j=0}^{\infty} j a_{ij} u^{j-1} (\alpha_1 n_1 + \ldots + \alpha_n n_n) + x_i \sum_j \zeta_{ij}(c)\, u^j$$

we find

$$\zeta_{i,j+1}(a) = \alpha_i (n_1 a_{1j} + \ldots + n_n a_{nj}) - j a_{ij} (\alpha_1 n_1 + \ldots + \alpha_n n_n), \quad j = 0, 1, \ldots$$

Hence, if we do not take the similarity transformation $(\alpha_1 = \ldots = \alpha_n = 1)$, into account, by the use of known standard procedures we find the following invariants:

$$a_{i1} = \text{Inv},\ i = 1, 2, 3, 4 \tag{3.2}$$

$$\omega_1 [n_3 (q^* - a_{31}q) + n_4 (q^* - a_{41}q)] - \omega_2 [n_1 (q^* - a_{11}q) + n_2 (q^* - a_{21}q)] = \text{Inv}$$

$$(\omega_1 = n_1a_{12} + n_2a_{22},\ \omega_2 = n_3a_{32} + n_4a_{42},\ q^* = n_1a_{11} + \ldots + n_na_{41})$$

Thus, for $h = 1$ and for one resonance relation (1.2), any analytic system of the fourth order can be reduced by a formal transformation to the form

$$x_i = x_i (\lambda_i + a_{i1}u + a_{i2}u^2)$$

where a_{i1} are fixed, while a_{i2} are related by the single condition (3.2).

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ON THE STABILITY OF MOTIONS OF CONSERVATIVE MECHANICAL SYSTEMS UNDER CONTINUALLY-ACTING PERTURBATIONS

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A. Ia. SAVCHENKO
(Donetsk)
(Received July 8, 1973)

We prove some theorems on the stability of motions of conservative mechanical systems under continually-acting perturbations, subject to specified constraints. In the investigation of stability of such type it is usually assumed only that the continually-acting perturbations are small [1]. Such a formulation omits from consideration an important class of conservative systems whose motions do not possess asymptotic stability because an integral invariant exists in them. However, in many problems concerning the structure of the continually-acting perturbations, certain information is available enabling us to estimate their influence